An Introduction to Qualitative Mathematical Modeling

Loop Analysis

or

Graphical Feedback Analysis
What is a Qualitative Mathematical Model?

- Graph theory
- E.g., The Familiar Influence Diagram
- Shows relational aspects of cause & effect: how variables interact.
Loop Analysis is much more powerful than an influence Diagram.

- Derived from Graph Theory

- But in addition to relational analysis
  - Depicts the effect of one variable to the next in terms of qualitative values of feedback.
  - Shows the direction of feedback paths through the community
If you can draw, you can Model!

Signed directed graphs. Arrows Indicate direction (+, 0, -) Qualitative signs denoting effect
The Making of a Loop Analysis Model (a Signed Digraph)

A Signed Diagraph depicts a community and its interactions

- It comprises nodes (variables) which represent members of the community.
- Community is linked by interactions among its members.
  - Pointed and blunt arrows show the direction of the effect of one variable on the next. (effect = feedback)
  - Curved arrows denote density-dependence &/or imply influences variables not implicitly stated in model.

Node + feedback - feedback Self Loops
Self Loops: density-dependent effects

- Negative self-loops self-regulate population growth (can mean its growth regulated by a community subsystem)
- Positive Feedback increases Feedback (e.g. numerical response to prey concentration...economic bubble)
Basic Community Interactions

- Predator-Prey
  - Mutualism
  - Commensalism
  - Amensalism

- Interference Competition
  - Exploitation Competition

- Microbial Culture
Their can be only one interaction link from one variable to the next.

Because the interaction link (coefficient) represents the net effect of one species upon the other.

Therefore aspects of competition & predation, etc. are combined in the sign
As a general rule, negative self-loops should be attached to most variables.
It represents a density-dependent self regulatory effect.
It may also imply that there are other variables that may be controlling its density but are not explicit in the model.
Remember that density-dependence among variables is expressed in material & energy transfer in the model.
Computer Does the Rest

- Transforms the digraph into a matrix
- Converts the signs of the interaction coefficients to cardinal values.
- Performs the matrix Algebra
- Provides estimates
  - stability of community structure
  - sign stability
  - robustness of model to differences in interaction strength (relative differences can be important)
Transformation of Signed Diagraph to a matrix

\[ a_j = \text{“source” species} \]
\[ a_i = \text{“target” species} \]

\[
\begin{bmatrix}
-a_{11} & -a_{12} \\
 a_{21} & -a_{22}
\end{bmatrix}
\]

effect of variable \( j \) on \( i \)
Numeric Substitutions for signed Coefficients

Reactors
= a_i

Actors = a_j

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<thead>
<tr>
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<th>Hare</th>
<th>Lynx</th>
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<tr>
<td>Hare</td>
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A tour of the Software

Go to

http://www.ent.orst.edu/loop/

Formal mathematical details to follow in next chapter
Build Signed Diagraph Using Powerplay

Loop Analysis

PowerPlay

If your web browser cannot display the PowerPlay Java applet and "Load Java Applet Failed..." appears in lower left side of browser, you need to install the Java Runtime Environment (JRE v1.5) which allows end-users to run Java applications. If you want to save the digraph you created, please use the Advanced Powerplay. If you need more detailed information about how to use PowerPlay, please click the menu link.

File  Data Tools  View  Help

Symbol pallette

LYNX  HARD  GRASS
Finish digraph then Click “Loop Analysis” button to get output.
Enter the community matrix (A):

```
n::=2
A::=array(1..n,1..n,[[[-1,1],[-1,-1]]]);[Lynx,Hare]
```

Community Matrix:

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Check Hurwitz Criteria for structural stability

Characteristic Polynomial:
\[
\begin{vmatrix}
1.00 & 2.00 & 2.00
\end{vmatrix}
\]

Characteristic Polynomial coefficients are all of the same sign. Pass Hurwitz Criterion I.

Hurwitz Matrix:
\[
\begin{bmatrix}
2 & 0 \\
1 & 2
\end{bmatrix}
\]

Hurwitz determinants:
- \( H_0 \): 2.00
- \( H_1 \): 4.00

All of the Hurwitz determinant are positive and greater than 0. Pass Hurwitz Criterion II. The system is stable.
Hurwitz Criteria I

- For local stability, all feedback must be negative. Because of sign conventions in the use of matrix algebra. The term “positive” is used to express this condition. (Criterion I)

- Feedback at different levels (size of loop as determined by the number of variables linked in a path) is calculated and listed as the coefficients of the “characteristic polynomial” (more later)
Qualitative Responses to Perturbation

$N^* + N$

$N^* = \text{pop'n at equilibrium}$

Perturbation Time

+$\text{feedback}$

$0\text{ feedback}$

$-\text{feedback}$
Hurwitz found that an over-reliance on safety valves that involved cycles containing many variables (higher level feedback) would lead to exploding engines.

Higher numbers of negative feedback should be distributed at lower levels (smaller pathways, smaller numbers of variables per cycle).

Cleverly, he used the coefficients of the characteristic polynomial to gain this insight.
Stability Metaphor

Local Stability

Global Stability

Neutral Stability = 0 feedback
1 variable
1 cycle or Loop

2 variables
2 cycles

NOTE: LOOP = PATH RETURNING TO ITS ORIGIN CROSSING A VARIABLE ONLY ONCE
ADJOINT, ABSOLUTE & WEIGHTED PREDICTIONS MATRICES

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Loop Analysis

n:=2;
A:=array(1..n,1..n,[-1,1],[-1,1]);
[Lynx,Hare]

**Adjoint Matrix:**

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**Absolute Feedback:**

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**Weighted Predictions:**

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**Adjoint Matrix:** The adjoint matrix is the transpose of the cofactor matrix. The cofactor matrix is the matrix of determinants of $a_{ij}$ multiplied by $-1^{i+j}$ (the matrix without the $i^{th}$ column and the $j^{th}$ row). The element of cofactor matrix $c_{ij}$ represents the effect of subsystems complementary to species $i$ and $j$. Complementary feedback is formed by subsystems of species that are not on the direct path between species $i$ and $j$, and is the product of disjunct loops. A positive effect to species $i$ (either increasing the birth rate or decreasing the death rate) is to read
The ADJOINT MATRIX comprises complementary feedback cycles & used to calculate (predict) the net effect of external input upon each and every member of the community. (e.g., what is the effect of liming acidified watersheds?)

The ABSOLUTE MATRIX accounts for the total number of both + and - Cycles (i.e. loops). It will be used to weight predictions (net effect). IF a new equilibrium value is predicted from the interaction of 55 (+) and 45 (-) cycles, the net is a change of 10 +. Is the net sufficient to predict an increase in standing crops from the external stimulus?

We use weighted values to determine the confidence of sign stability for each community member (Next slide)
Weighted Matrix

- The WEIGHTED MATRIX is calculated by dividing each variable in the ADJOINT MATRIX by its representative in the ABSOLUTE MATRIX.

- In this example the variable in question has a weighted value of \((55-45)/100 = (0.1)\) & from our simulations unlikely to be Sign Stable.

- This then is a measure of predictive reliability.
The prediction can be red down the column!
Predictions of Negative Input(s)

- Trapping Lynx decreases lynx & increases hare.
- Trapping Hares for fur decreases both critters.
- Trapping both species really decreases Lynx (-2) but doesn’t affect Hares (+1 -1= 0).
- You can add across the columns on the same row to get the joint effect of two inputs. Great heh?

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Test of Robustness

Loop Analysis

Strengths of Interactions matter. Read next slide
Strengths of Interaction

- Some topologies (structural properties) of communities can become stable or unstable depending upon the relative differences in the values of the interactive coefficients among “species”.

- Some model communities that pass both Hurwitz Criteria given Cardinal values will fail one or both criteria when randomly assigned Ordinal values of different strengths.

- This is a test of robustness & reliability of stability and predictive outcome of qualitative models.

- Run this test several times to get statistical values for models prone to fail.
Check out published webs

- You can run many published models already loaded into the freeware.
- To test how the behavior of models are affected by self-loops you have the option of just putting self-loops on the base variables of the food web or in addition, putting self-loops on consumers as well.